Exercise 2.1.4

(Exact solution of $\dot{x} = \sin x$) As shown in the text, $\dot{x} = \sin x$ has the solution $t = \ln|(\csc x_0 + \cot x_0)/(\csc x + \cot x)|$, where $x_0 = x(0)$ is the initial value of x.

a) Given the specific initial condition $x_0 = \pi/4$, show that the solution above can be inverted to obtain

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).$$

Conclude that $x(t) \to \pi$ as $t \to \infty$, as claimed in Section 2.1. (You need to be good with trigonometric identities to solve this problem.)

b) Try to find the analytical solution for $x(t)$, given an *arbitrary* initial condition x_0.

Solution

If $x_0 = \pi/4$, then the solution to $\dot{x} = \sin x$ reduces to

$$t = \ln\left|\frac{\csc\frac{\pi}{4} + \cot\frac{\pi}{4}}{\csc x + \cot x}\right|$$

$$t = \ln\left|\frac{\sqrt{2}+1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right|$$

$$t = \ln\left|\frac{\sqrt{2}+1}{1} \cdot \frac{\sin x}{1+\cos x}\right|$$

$$t = \ln\left|\frac{\sqrt{2}+1}{1} \cdot \tan\frac{x}{2}\right|.$$

Exponentiate both sides.

$$e^t = \left|\frac{\sqrt{2}+1}{1} \cdot \tan\frac{x}{2}\right|$$

Remove the absolute value sign on the right side by placing $\pm$ on the left side.

$$\pm e^t = \frac{\sqrt{2}+1}{1} \cdot \tan\frac{x}{2}$$

Solve for x.

$$\tan\frac{x}{2} = \pm\frac{e^t}{1+\sqrt{2}}$$

$$\frac{x}{2} = \tan^{-1}\left(\pm\frac{e^t}{1+\sqrt{2}}\right) + n\pi, \quad n = 0, \pm1, \pm2, \ldots$$

$$x(t) = \pm 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) + 2n\pi$$

Plugging in $t = 0$ results in $\pi/4$ only if the plus sign is chosen and $n = 0$. Therefore,

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = 2\tan^{-1}(\infty) = 2\left(\frac{\pi}{2}\right) = \pi.$$

Here the aim is to solve the following initial value problem.

$$\frac{dx}{dt} = \sin x, \quad x(0) = x_0$$

Divide both sides by $\sin x$.

$$\frac{\frac{dx}{dt}}{\sin x} = 1$$

Rewrite the left side using the chain rule.

$$\frac{d}{dt} \ln \left| \tan \frac{x}{2} \right| = 1$$

The absolute value sign is included because the logarithm argument cannot be negative. Integrate both sides with respect to t.

$$\ln \left| \tan \frac{x}{2} \right| = t + C$$

Apply the initial condition $x(0) = x_0$ now to determine C.

$$\ln \left| \tan \frac{x_0}{2} \right| = C$$

As a result, the previous equation becomes

$$\ln \left| \tan \frac{x}{2} \right| = t + \ln \left| \tan \frac{x_0}{2} \right|.$$

Bring the two logarithms to the left side and combine them.

$$\ln \left| \frac{\tan \frac{x}{2}}{\tan \frac{x_0}{2}} \right| = t$$

Exponentiate both sides.

$$\left| \frac{\tan \frac{x}{2}}{\tan \frac{x_0}{2}} \right| = e^t$$

Place $\pm$ on the right side in order to remove the absolute value sign on the left side.

$$\frac{\tan \frac{x}{2}}{\tan \frac{x_0}{2}} = \pm e^t$$

Solve for x.

$$\begin{aligned}
\tan \frac{x}{2} &= (\pm e^t) \tan \frac{x_0}{2} \\
\frac{x}{2} &= \tan^{-1} \left[(\pm e^t) \tan \frac{x_0}{2} \right] + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \\
x(t) &= \pm 2 \tan^{-1} \left(e^t \tan \frac{x_0}{2} \right) + 2n\pi
\end{aligned}$$

Plugging in $t = 0$ results in x_0 only if the plus sign is chosen and $n = 0$. Therefore,

$$x(t) = 2 \tan^{-1} \left(e^t \tan \frac{x_0}{2} \right).$$